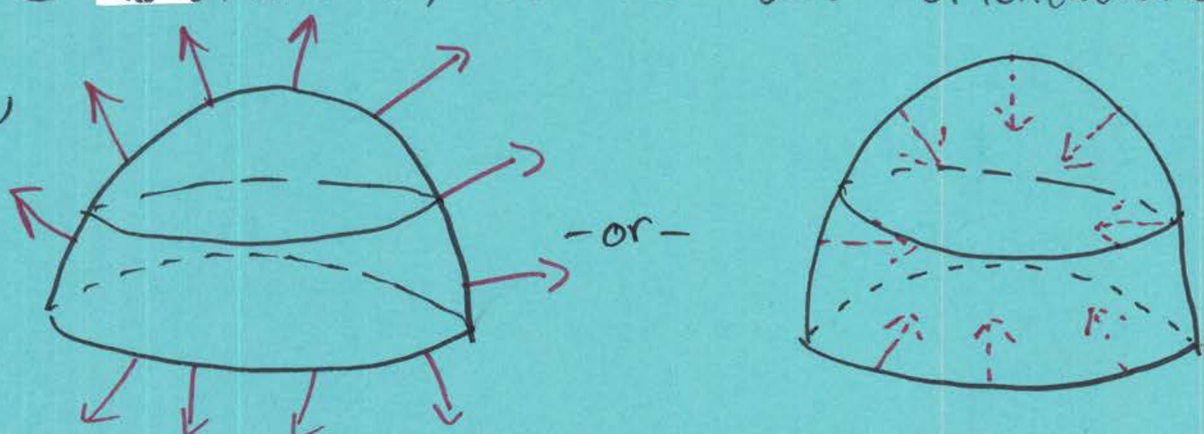
Lecture 41

Orientations of Surfaces

For the remainder of the material in this class, it will be important that our surfaces be oriented: Non-oriented surfaces have pathological behavior (c.f., the Möbius band). For our purposes, an orientation on a surface S is a choice of a continuous unit normal vector field on S.

If S is orientable, it has two orientations:

e.g.,



one for each side of the surface. In particular, if the orientations are $\vec{n}_1$ and $\vec{n}_2$, then $\vec{n}_1 = -\vec{n}_2$.

So, how do we find orientations?

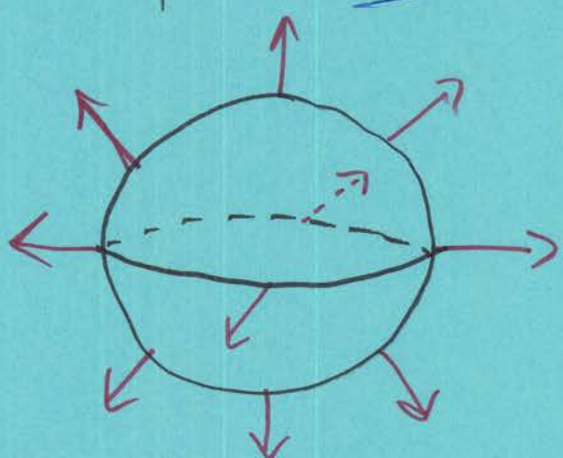
If S has parametrization: $\vec{r}(u,v)$, then the orientations on S are given by:

$$\boxed{\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \quad \text{-or-} \quad -\vec{n} = \frac{\vec{r}_v \times \vec{r}_u}{|\vec{r}_v \times \vec{r}_u|}}$$

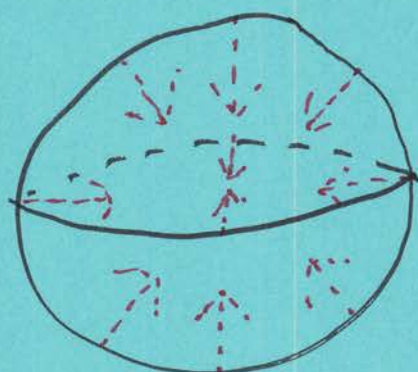
(Remember $\vec{r}_u \times \vec{r}_v$ is perpendicular to S?)

Notice the order in the cross product matters now!

Now, if we have a closed surface, that is a surface which is the boundary of a solid (e.g. a sphere, torus, ellipsoid, box, etc.), we call the orientation which points outward the positive orientation.



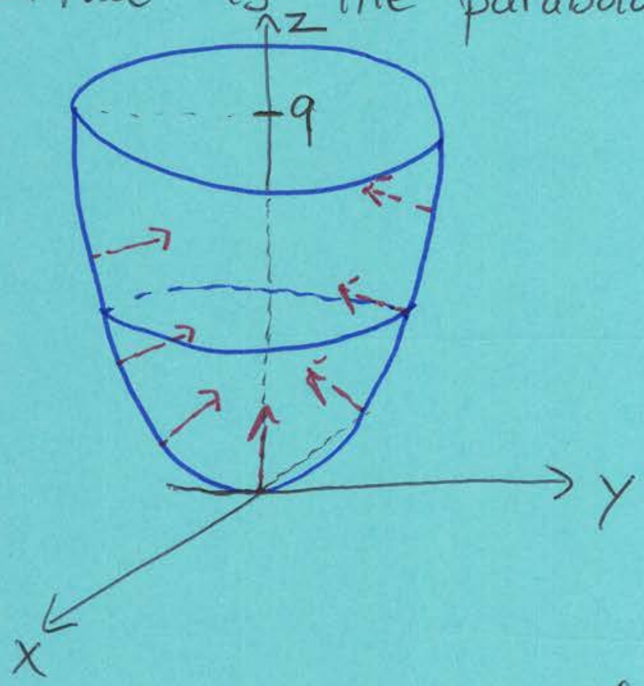
Positive Orientation



Negative Orientation

<u>Ex</u>: Find the upward pointing orientation on the surface which is the graph of $f(x,y)=x^2+y^2$, where $x^2+y^2\le 9$.

<u>Sol</u>: The surface is the paraboloid:



and the orientation we want is the indicated one.

Let's parametrize this: (with cylindrical)

$$\vec{r}(r,\theta)=\langle r\cos\theta, r\sin\theta, r\rangle,\quad r\le 3,\ 0\le\theta\le 2\pi.$$

$$\vec{r}_r=\langle \cos\theta, \sin\theta, 1\rangle,\quad \vec{r}_\theta=\langle -r\sin\theta, r\cos\theta, 0\rangle.$$

Let's try $\vec{r}_\theta\times\vec{r}_r$:

$$\vec{r}_\theta\times\vec{r}_r=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -r\sin\theta & r\cos\theta & 0\\ \cos\theta & \sin\theta & 1\end{vmatrix}=\langle r\cos\theta, r\sin\theta, -r\rangle$$

Now, test this at a point (we can test before normalizing)

say, $(r,\theta) = (1, \frac{\pi}{2}) \Rightarrow \vec{r}(1, \frac{\pi}{2}) = \langle 0, 1, 1 \rangle$

$\vec{r}_\theta \times \vec{r}_r (1, \frac{\pi}{2}) = \langle 0, 1, -1 \rangle$.

Does $\langle 0,1,-1 \rangle$ point in the right direction from the point $(0,1,1)$ on S? NO! So, we actually wanted the orientation coming from $\vec{r}_r \times \vec{r}_\theta$. $(= -\vec{r}_\theta \times \vec{r}_r)$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}\, r$$

$\vec{r}_r \times \vec{r}_\theta = -\vec{r}_\theta \times \vec{r}_r$, so our orientation is:

$$\vec{n} = \frac{\vec{r}_r \times \vec{r}_\theta}{|\vec{r}_r \times \vec{r}_\theta|} = \frac{1}{\sqrt{2}\, r}\langle -r\cos\theta, -r\sin\theta, r \rangle = \frac{1}{\sqrt{2}}\langle -\cos\theta, -\sin\theta, 1 \rangle$$

$\diamond$

Flux

Flux, as a concept, measures the rate at which something flows through a surface. For example, if $\vec{E}$ is an electric field, and Σ is a closed, positively oriented surface, then the electric flux of $\vec{E}$ through Σ

is given by <u>Gauss' law</u> as

$$\Phi_E = \frac{Q_{net}}{\varepsilon_0} \quad (Q_{net} = \text{total charge inside } \Sigma)$$

The flux across a tiny patch of the surface is $\vec{E}\cdot\vec{n}\,\Delta S_{ij}$ ($\vec{n}$ is a choice of orientation on the surface). So, as you might expect, the <u>total flux</u> is

$$\text{flux} = \iint_\Sigma \vec{E}\cdot\vec{n}\,dS = \iint_\Sigma \vec{E}\cdot d\vec{S} = \Phi_E$$

We set $\vec{n}\,dS = d\vec{S}$.

If Σ is parametrized by $\vec{r}(u,v)$, $(u,v)\in D$, then (with $\vec{n} = \vec{r}_u \times \vec{r}_v$)

$$d\vec{S} = \vec{n}\,dS = \frac{\vec{r}_u\times\vec{r}_v}{\cancel{|\vec{r}_u\times\vec{r}_v|}}\cancel{|\vec{r}_u\times\vec{r}_v|}\,dA = (\vec{r}_u\times\vec{r}_v)\,dA$$

Let's sum this all up:

<u>Def</u>: If $\vec{F}$ is a continuous vector field defined on a surface S which has orientation $\vec{n}$, then the <u>flux of $\vec{F}$ across S</u> (or <u>surface integral of $\vec{F}$ over S</u>, or <u>vector surface integral</u>) is:

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS}$$

Moreover, if S is parametrized by $\vec{r}(u,v)$, and $\left((u,v)\in D\right)$

if $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$, then

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\, dA}$$

Ex: Use Gauss' law to find the charge enclosed by the unit sphere centered at $(0,0,0)$ if the electric field is $\vec{E} = \langle x, y, z\rangle$.

Sol: First, parametrize the surface, S:

$\vec{r}(\theta,\varphi) = \langle \cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi\rangle$, $0\le\theta\le 2\pi$, $0\le\varphi\le\pi$

Now, Gauss' law requires S to have positive (outward) orientation, so let's figure out the order for the cross product:

$\vec{r}_\theta = \langle -\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0\rangle$

$\vec{r}_\varphi = \langle \cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi\rangle$

If we try

$\vec{r}_\theta \times \vec{r}_\varphi = -\sin\varphi\langle \cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi\rangle$

Check this at a point, say $\vec{r}(\pi, \frac{\pi}{2}) = \langle -1, 0, 0 \rangle$

$$\vec{r}_\theta \times \vec{r}_\varphi(\pi, \tfrac{\pi}{2}) = -\sin\tfrac{\pi}{2}\langle \cos\pi \sin\tfrac{\pi}{2}, \sin\pi \sin\tfrac{\pi}{2}, \cos\tfrac{\pi}{2}\rangle$$

$$= (-1)\langle -1, 0, 0\rangle = \langle 1, 0, 0\rangle,$$

which points <u>inward</u>, so, the wrong way!

So, $\vec{r}_\varphi \times \vec{r}_\theta$ is the choice consistant with the orientation.

$$\vec{r}_\varphi \times \vec{r}_\theta = -\vec{r}_\theta \times \vec{r}_\varphi = \sin\varphi\langle \cos\theta \sin\varphi, \sin\theta\sin\varphi, \cos\varphi\rangle.$$

Finally, we compute the flux

$$\iint_S \vec{E}\cdot d\vec{S} = \iint_D \vec{E}(\vec{r}(\theta,\varphi))\cdot(\vec{r}_\theta \times \vec{r}_\varphi)\, dA$$

$$= \int_0^\pi\int_0^{2\pi} \sin\varphi\Big(\underbrace{\underbrace{\cos^2\theta\sin^2\varphi + \sin^2\theta\sin^2\varphi}_{\sin^2\varphi} + \cos^2\varphi}_{1}\Big)\, d\theta\, d\varphi$$

$$= \int_0^\pi\int_0^{2\pi} \sin\varphi\, d\theta\, d\varphi = 2\pi\int_0^\pi \sin\varphi\, d\varphi = 2\pi(-\cos\varphi)\Big|_0^\pi$$

$$= 2\pi(-(-1)-(-1)) = 4\pi = \Phi_E = \frac{Q}{\varepsilon_0}$$

So, the charge is $Q = 4\pi\varepsilon_0$

$\diamond$

Ex: Compute $\iint_S (\text{curl } \vec{F}) \cdot d\vec{S}$ where $\vec{F} = \langle -2yz, y, 3x \rangle$ and S is the piece of the paraboloid $z = 5 - x^2 - y^2$ above the plane $z = 1$, with upward orientation.

Sol: First, parametrize S:

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 5 - r^2 \rangle, \; 0 \le r \le 2, \; 0 \le \theta \le 2\pi$$

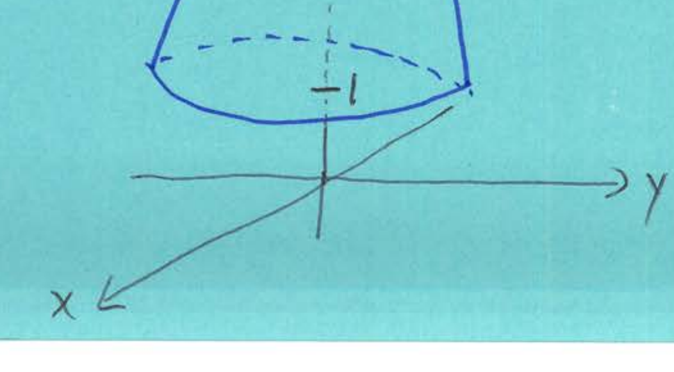
$$\vec{r}_r = \langle \cos\theta, \sin\theta, -2r \rangle, \; \vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

Try, for the orientation

$$\vec{r}_r \times \vec{r}_\theta = \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle$$

Check this at, say, $\vec{r}(1,0) = \langle 1, 0, 4 \rangle$

$$\vec{r}_r \times \vec{r}_\theta (1,0) = \langle 2, 0, 1 \rangle$$



So, this was the correct choice.

Now, $\operatorname{curl} \vec{F} = \langle 0, -2y-3, 2z \rangle$

$$(\operatorname{curl} \vec{F})(\vec{r}(r,\theta)) = \langle 0, -2r\sin\theta - 3, 10 - 2r^2 \rangle$$

$$(\operatorname{curl} \vec{F})(\vec{r}(r,\theta)) \cdot (\vec{r}_r \times \vec{r}_\theta) = 0 - 4r^3\sin^2\theta - 6r^2\sin\theta + 10r - 2r^3$$

So,

$$\iint_S (\operatorname{curl}\vec{F}) \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 \left(-4r^3\sin^2\theta - 6r^2\sin\theta + 10r - 2r^3\right) dr\, d\theta$$

$$= \int_0^{2\pi} \left(-r^4\sin^2\theta - 2r^3\sin\theta + 5r^2 - \tfrac{1}{2}r^4 \right)\Big|_0^2 \, d\theta$$

$$= \int_0^{2\pi} \left(\underline{-16\sin^2\theta} - 16\sin\theta + 20 - 8 \right) d\theta \qquad \left(\sin^2\theta = \frac{1-\cos 2\theta}{2} \right)$$

$$= \int_0^{2\pi} \left(-8 + 8\cos 2\theta - 16\sin\theta + 12 \right) d\theta = \int_0^{2\pi} \left(8\cos 2\theta - 16\sin\theta + 4 \right) d\theta$$

$$= \left(4\sin 2\theta + 16\cos\theta + 4\theta \right)\Big|_0^{2\pi} = (0 + 16 + 8\pi) - (0 + 16 + 0) = 8\pi$$

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